Complex Analysis Preliminary Exam

January 14, 2023

1. Find all complex solutions of the equation

$$z^6 = -8.$$

Your answer should be simplified and should be in the form $x + iy$ with $x, y \in \mathbb{R}$.

2. Prove that there is no function f such that f is analytic on the punctured unit disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and f' has a simple pole at 0.

3. (a) For $R > 0$, find a conformal map from $S_R = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < R\}$ to the open unit disk.

   (b) Prove that if f is entire and the imaginary part of f is bounded on the whole complex plane, then f must be constant.

4. Find

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(1+x^2)^2} dx.$$

5. Let

$$h(z) = \frac{11}{2z^2 + 9z - 5}.$$

Find the Laurent expansion for h centered at 0 which converges at the point $z = -3i$, and state precisely where this Laurent series converges.

6. Let D be the open disk defined by $|z - 3i| < 2$. Find

$$\int_{\partial D} f(z)\, dz,$$

where $f(z) = \dfrac{2z-1}{z^2 - z + 2}$, where ∂D is oriented counterclockwise,

   (a) using the Cauchy integral formula.

   (b) using the Argument Principle.

7. Find a holomorphic function f on the disk defined by $|z - 1| < 1$ that satisfies

$$f\left(\frac{n}{n+1}\right) = 1 - \frac{1}{n(n+1)}$$

for all $n = 2, 3, \ldots$ Prove or disprove that there exists a different holomorphic function with the same property.

8. Let ϕ be the entire function defined by $\phi(z) = e^{iz} + 10z + 2$ for $z \in \mathbb{C}$.

   (a) Let B denote the open disk of radius 1 centered at $3 - 4i \in \mathbb{C}$. Prove that there exists $z_0 \in \mathbb{C}$ such that $|z_0 - 3 + 4i| = 2$ and $|\phi(w)| < |\phi(z_0)|$ for all $w \in B$.

   (b) Let Δ denote the open disk of radius 1 centered at $0 \in \mathbb{C}$. Prove that there exists $z_1 \in \Delta$ such that $\phi(z_1) = 0$.